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BICIRCULAR GEOMETRY AND THE LATTICE OF FORESTS OF A GRAPH

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Dedicated to Professor Fred Supnick

LET us order the set $\mathcal{F}$ of all (not necessarily spanning) forests in a graph Γ in the following way: $F' \geq F$ if F' is constructible by taking some (or none) of the trees of F and (optionally) adding edges which link some of these trees. Then $\mathcal{F}$ is a geometric lattice; its rank is the number n of nodes of Γ; its characteristic polynomial (when Γ is finite) equals $(-1)^n f_\Gamma(1-\lambda)$ where $f_\Gamma(x) = \sum f_k(\Gamma)x^{n-k}$, $f_k(\Gamma)$ = the number of k-tree spanning forests in Γ; its Möbius invariant equals $(-1)^n \sum f_k(\Gamma)$. The meet and join of $F_1 = (X_1, E_1)$ and $F_2 = (X_2, E_2)$ are: $F_1 \wedge F_2$ = the forest consisting of $F_1 \cap F_2$ and the subgraph of $F_1 \cup F_2$ induced on $(X_1 \setminus X_2) \cup (X_2 \setminus X_1)$; and $F_1 \vee F_2$ = the forest composed of those trees of $F_1 \cup F_2$ that lie in $X_1 \cap X_2$. And more: see Theorems 7 and 8.

These facts belong to what we call "bicircular geometry": the study of the bicircular matroid both abstractly and in coordinatized representations. For $\mathcal{F}$ with the stated ordering is essentially the lattice of flats of the bicircular matroid of Γ° (Γ with a loop at every node) and its properties are nice versions of those of bicircular matroids in general. The latter in turn are the extreme examples of voltage-graphic matroids opposite in a sense to the usual graphic matroids; so we can draw on the general theory developed for voltage graphs and biased graphs in [**14–17**].

The necessary structure theory for voltage graphs we sketch in Section 1. In Section 2 we obtain our general results on bicircular matroids and the forest lattice, including their chromatic and Tutte invariants. With this theory in hand, we study in Section 3 the concrete side of bicircular geometry: the maximal dissections of real linear and affine spaces by arrangements of hyperplanes with "two-term" equations. In the linear space $\mathbb{R}^n$ these are equations of the forms $x_j = ax_i$ and $x_i = 0$. We determine first the number of parts into which $\mathbb{R}^n$ is dissected by any such arrangement. The largest possible number, given how many equations involve each pair or singleton of variables, is a chromatic invariant of the bicircular matroid. Then we admit also inhomogeneous or "affine" two-term equations, whose form is $x_i = a$. Finally we consider arrangements of hyperplanes whose equations are two-term in barycentric coordinates, extending and explaining previous work on "apical" dissections of a simplex [**13**].

1. Graphs, voltage, and bias

We consider a graph $\Gamma=(N, E)$, whose node set is N (of cardinality n) and whose edge set is E. The graph may be infinite, may have multiple edges, and may have besides links also loops (two coincident endpoints) and free loops (no endpoints; they are required for technical reasons).[1] We distinguish between the two senses of an edge: e traversed in the opposite direction we denote e^{-1}; but we do not orient e by preferring either sense. If $X \subseteq N$, we denote the induced subgraph on X by $\Gamma{:}\ X$, the complement $N \backslash X$ by X^c; we call X *stable* if $\Gamma{:}\ X$ has no edges. If $\Delta=(X, D)$ is a subgraph of Γ and $S \subseteq E$, the *restriction* $\Delta \mid S$ is $(X, S \cap D)$. We write $N(S)$ for the set of all endpoints of edges in $S \subseteq E$; $\mathscr{C}$ for the set of circles of Γ (simple closed paths, not including free loops); $c(\Gamma)$ for the number of connected components of Γ (meaning *node components*: an isolated node counts but a free loop does not); $t(\Gamma)$ for the number of tree components; Γ_t for the subgraph consisting of the tree components; and $N_u(\Gamma)$ for the set of nodes of nontree components. Sometimes we call Γ an *ordinary graph* to distinguish it from a voltage graph or biased graph.[2] The ordinary *graphic matroid* of Γ, whose circuits are the circles and free loops, we denote $G(\Gamma)$. The corank in $G(\Gamma)$ of $S \subseteq E$ is $c(N, S)-c(\Gamma)$ and its rank is $\operatorname{rk} S=n-c(N, S)$, when N is finite. In general the corank is $\sum_\Delta [c(\Delta \mid S)-1]$ summed over the components of Γ.

A *bicycle* is a minimal connected graph with two independent circles: either a theta graph or a *handcuff* (two circles which meet at a single node, or two disjoint circles and a simple connecting path meeting the circles only at its endpoints). Taking for circuits the bicycles and free loops of Γ we have a matroid on E; this is the *bicircular matroid* of Γ, which we write $B(\Gamma)$. The bicircular matroid was noted by Klee ([**4**], pp. 145–46) in an infinitary version for infinite graphs and was independently discovered by Simões-Pereira [**9**] in the finitary version we treat here.

A *voltage graph* $\Phi=(\Gamma, \varphi)$ is a graph Γ together with a *voltage mapping* φ from E to a group such that $\varphi(e)=1$ for every free loop e. We regard φ as directional, meaning that $\varphi(e^{-1})=\varphi(e)^{-1}$. A circle $e_1 e_2 \cdots e_r$ is called *balanced* if $\varphi(e_1)\varphi(e_2)\cdots\varphi(e_r)=1$; the class of balanced circles is $\mathscr{B}(\varphi)$. A graph is balanced if all its circles are. The number of balanced components of a subgraph (N, S) is denoted $b(S)$. The *voltage-graphic*

[1] The half edges which appear in [**14**], [**15**], etc. are here replaced by unbalanced loops, to which they are equivalent in all matroidal questions.

[2] Our definitions of $\mathscr{C}$ and "ordinary" differ trivially from those in [**14**], [**15**], etc.

matroid $G(\Phi) = G(\Gamma, \varphi)$ is the matroid on E whose circuits are the balanced circles, the free loops, and the bicycles in which no circle is balanced. (See **[15]**, Sections 5 and 9, for properties and proofs.) Because this matroid depends not on φ itself but only on $\mathcal{B} = \mathcal{B}(\varphi)$, we also write $G(\Gamma, \mathcal{B})$, the matroid of $(\Gamma, \mathcal{B})$ (called a *biased graph*; cf. **[14]** for the full definition).

If for example $\varphi \equiv 1$, then $\mathcal{B}(\varphi) = \mathcal{C}$ and we have $G(\Gamma, \mathcal{C})$, which is clearly the graphic matroid $G(\Gamma)$.

At the opposite extreme if $\mathcal{B} = \varnothing$ we have the *totally biased graph* $(\Gamma, \varnothing)$, whose matroid is the bicircular matroid. One voltage which yields $\mathcal{B}(\varphi) = \varnothing$ is $\varphi(e) = e$ with the voltage group the free group generated by E. A more interesting voltage—because it leads in the finite case to an interpretation of the characteristic polynomial of $G(\Gamma, \varnothing)$—is the *binary voltage* $\varphi_2(e) = e$ with voltage group the binary vector space with basis E, that is $\mathbb{Z}_2^E$. (A smaller voltage group is obtained by factoring out the subspace spanned by the edge set of a maximal forest.)

2. The bicircular matroid and the forest lattice

2a. STRUCTURE OF THE BICIRCULAR MATROID. Our first theorem is a multiple characterization of the bicircular matroid.

THEOREM 1. *The bicircular matroid* $B(\Gamma)$ *of a graph* $\Gamma = (N, E)$ *has rank* $n - t(\Gamma)$, *where* $n = |N|$. *It is determined by any one of the following properties:*

(a) *The closure of an edge set S is*

$$\operatorname{clos} S = S \cup \{e \in E: \text{all endpoints of } e \text{ are in nontree components of } (N, S)\}.$$

(b) *An edge set is closed* $\Leftrightarrow$ *it consists of an induced subgraph, disjoint trees, and the free loops.*

(c) *An edge set S is independent* $\Leftrightarrow$ *it contains no free loops and each component of* (N, S) *is a tree plus at most one more edge.*

(d) *An edge set is dependent* $\Leftrightarrow$ *it contains two circles connected within it or it contains a free loop.*

(e) *An edge set is a circuit* $\Leftrightarrow$ *it is a bicycle or a free loop.*

(f) *An edge set spans* $\Leftrightarrow$ *it contains every tree component of* Γ *but no other component of it is a tree.*

(g) *An edge set S is a basis* $\Leftrightarrow$ *it contains every tree component of* Γ, *and every other component of* (N, S) *is a tree plus one more edge.*

(h) *An edge set H is a coatom* $\Leftrightarrow$ *either* $H = E \setminus \{e\}$ *where e lies in* Γ_t, *or else H has one tree component T outside* Γ_t *and H contains* $E{:}\,N(T)^c$ *and all free loops.*

(i) *An edge set is a bond of* $B(\Gamma) \Leftrightarrow$ *it is a minimal set whose removal increases the number of tree components.*

(j) *The rank of an edge set S is*

$$\operatorname{rk} S = |N_u(S)| + \sum_T |E(T)|,$$

summed over all tree components T of S; when N is finite this reduces to

$$\operatorname{rk} S = n - t(N, S).$$

(j′) *The rank of an edge set S is*

$$\operatorname{rk} S = \min_{R \subseteq S} (|N(R)| + |S \setminus R|).$$

(k) *The corank of an edge set S is* $t(N, S) - t(\Gamma)$ *if N is finite. In general it is*

$$t((N, S) \setminus \Gamma_t) + \sum_T [t(T \mid S) - 1]$$

summed over the tree components of Γ.

Proof. As we saw in Section 1, $B(\Gamma)$ is voltage graphic. Parts (a)–(j) are special cases of the existence and description theorem for voltage-graphic matroids in [**15**] (see Theorem 5.1 and Section 9). Part (e), being the original definition of $B(\Gamma)$, shows that the voltage-graphic matroid guaranteed by the cited results is the right matroid.

Part (j′) is an immediate consequence of (c) and a theorem of Edmonds and Rota on submodular set functions [**3**] (see [**10**], Corollary 8.1.1). It was proved by Matthews in the course of showing that $B(\Gamma)$ is transversal ([**7**], proof of Theorem 3.1).

Part (k) is readily verified from (h).

Note that $B(\Gamma)$ as we define it is the finitary bicircular matroid, Example 1.3 in [**8**], not the infinitary version in Example 1.5 of [**8**]. To get the latter one must replace the phrase "tree component" throughout by "finite tree component" and make a few other modifications to handle the fact that, in the infinitary bicircular matroid, a one-way infinite path behaves matroidally like an unbalanced circle.

The next result describes the minors of $B(\Gamma)$. Let Γ_S be the graph obtained from Γ by contracting each tree component of S to a point, deleting each nontree component (both nodes and edges), and replacing each link e from v to w, where $e \notin S$ and w lies in a nontree component of S, by a loop[3] at v if v lies in a tree component of S, a free loop otherwise.

[3] The loop at v is more properly viewed as a half edge (cf. [**14**], [**15**]), but we can ignore the difference.

THEOREM 2. *Let $S \subseteq E$ in the graph Γ. Then the restriction of $B(\Gamma)$ to S is*

$$B(\Gamma) \mid S = B(N, S)$$

and the contraction by S is

$$B(\Gamma)/S = B(\Gamma_S).$$

The restriction part is obvious. The contraction formula is a special case of the description of contractions of voltage graphs in [**15**]. If S is finite we can also get the result from Matthews' constructions of one-edge minors in [**7**].

Let us consider the flats of $B(\Gamma)$ more closely. (We shall ignore free loops). To describe a flat A it suffices to name its tree components, say $T_1, T_2, \ldots$. For if $X = N(T_1 \cup T_2 \cup \cdots)$, then $A = T_1 \cup T_2 \cup \cdots \cup (E{:}\ X^c)$. On the other hand suppose $T_1, T_2, \ldots$ form a forest with node set X: then $T_1 \cup T_2 \cup \cdots \cup (E{:}\ X^c)$ is a flat, although it corresponds to the forest $T_1 \cup T_2 \cup \cdots$ only if $\Gamma{:}\ X^c$ has no tree components. This gives us an intrinsic description of the flats of $B(\Gamma)$.

THEOREM 3. *The lattice of flats of $B(\Gamma)$ is isomorphic to the set $\mathcal{F}_0$ of all forests $F \subseteq \Gamma$ such that $\Gamma{:}\ N(F)^c$ has no tree components, ordered by $F' \geq F$ if F' consists of some trees of F (or no trees, if $N(F') = \varnothing$) plus (possibly) linking edges.*

The 0 element is $(N, \varnothing)$, the 1 element is Γ_t, the atoms are $(N, \{e\})$ for each link e and $(N\backslash\{v\}, \varnothing)$ for each vertex v which supports a loop, and the coatoms are $\Gamma_t \backslash \{e\}$ for each $e \in E(\Gamma_t)$ and $\Gamma_t \cup T$ for each tree $T \subseteq \Gamma \backslash \Gamma_t$ such that $\Gamma{:}\ N(T)^c$ has no tree components outside Γ_t. The rank of a forest F is $|E(F)| + |N(F)^c|$, which when N is finite equals $n - t(F)$. The corank of F equals $t(F) - t(\Gamma)$ if N is finite, and in general

$$t(F\backslash\Gamma_t) + \sum_T [t(T \cap F) - 1]$$

summed over the tree components of Γ.

Completion of proof. The ordering is easily deduced from the fact that a forest F corresponds to the edge set

$$E(F) \cup (E{:}\ N(F)^c) \cup \{\text{free loops}\}.$$

The rank and corank functions follow from Theorem 1 similarly.

2b. TUTTE INVARIANTS AND PROPER COLORING. Here we assume finiteness of the graphs and voltage groups. The *chromatic polynomial* $\chi_\Phi(\lambda)$ of the voltage graph $\Phi = (\Gamma, \varphi)$ may be defined by the formula

$$\chi_\Phi(\lambda) = \sum_A \mu_\Phi(\varnothing, A)\lambda^{b(A)} = \lambda^{b(\Phi)} p_{G(\Phi)}(\lambda), \tag{1}$$

where the sum is taken over flats of $G(\Phi)$, μ_Φ is the Mobius function of $G(\Phi)$, and $p_G(\lambda)$ denotes the characteristic polynomial of G. Recall the convention that $\mu(\varnothing, A)\equiv 0$ if $\varnothing$ is not a flat.

The chromatic polynomial counts proper colorings of Φ. Let g be the order of the voltage group $\mathfrak{G}$, and for each integer $\mu \geqslant 0$ let

$$K_\mu^* = \{1, 2, \ldots, \mu\} \times \mathfrak{G},$$

called the *zero-free μ-color set*, and

$$K_\mu = \{0\} \cup K_\mu^*,$$

the *μ-color set* for $\mathfrak{G}$. The voltage group has the right action $(i, \alpha)\gamma = (i, \alpha\gamma)$ and $0\gamma = 0$. A *(zero-free) coloring of Φ in μ colors* is a mapping $\kappa\colon N \to K_\mu$ (to K_μ^* if zero-free); it is *proper* if, for each edge e from v to w,

$$\kappa(w) \neq \kappa(v)\varphi(e).$$

The number of proper colorings equals $\chi_\Phi(\mu g + 1)$. The number of zero-free proper colorings equals $\chi_\Phi^b(\mu g)$, where

$$\chi_\Phi^b(\lambda) = \sum_A \mu_\Phi(\varnothing, A)\lambda^{c(A)}$$

summed over balanced flats of $G(\Phi)$. (This material is drawn from [**16**].)

The fundamental theorem, which makes possible computation of $\chi_\Phi(\lambda)$, is the balanced expansion formula ([**17**], Theorem 1.1 and Appendix):

$$\chi_\Phi(\lambda + 1) = 0^{l_0} \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} \chi_{\Phi:X}^b(\lambda), \tag{2}$$

where Φ: X means the voltage graph induced on X and l_0 is the number of free loops. Generally it is easier to calculate balanced than unbalanced polynomials; certainly that is so for totally biased graphs. Indeed in $G(\Gamma, \varnothing)$ the balanced flats are just the spanning forests (ignoring the free loops, if any, which are in every flat). Thus if there are no free loops,

$$\chi_{(\Gamma, \varnothing)}^b(\lambda) = (-1)^n \sum_{k=0}^{n} f_k(\Gamma)(-\lambda)^k, \tag{3b}$$

so that

$$\chi_{(\Gamma, \varnothing)}(\lambda) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X|} \sum_{k=0}^{|X|} f_k(\Gamma\colon X)(1-\lambda)^k. \tag{3u}$$

Other important polynomials of a voltage graph are the *Whitney* *polynomial,*

$$w_\Phi(x,\lambda)=\sum_A x^{\operatorname{rk} A}\chi_{\Phi/A}(\lambda),$$

whose coefficient of x^i is the number of colorings of Φ in $(\lambda-1)/g$ colors whose set of improperly colored edges has rank i in $G(\Phi)$, and its balanced relative

$$w^b_\Phi(x,\lambda)=\sum_{A \text{ balanced}} x^{\operatorname{rk} A}\chi^b_{\Phi/A}(\lambda),$$

both sums ranging over flats of $G(\Phi)$. (See **[16]** for the interpretations of the Whitney polynomials.) Also the *dichromatic polynomial*

$$Q_\Phi(u,v)=\sum_{S\subseteq E} u^{b(S)}v^{|S|-\operatorname{rk} S},$$

and its balanced relative $Q^b_\Phi(u,v)$ in which S is restricted to balanced edge sets (for the dichromatic polynomials see **[14]**). These polynomials have the following properties. First,

$$\chi_\Phi(\lambda)=(-1)^nQ_\Phi(-\lambda,-1)=0^{l_0}w_\Phi(0,\lambda)$$

and balanced cognate. The coefficient of $x^i\lambda^{n-i}$ in $w_\Phi(x,\lambda)$ equals the Whitney number of the second kind, $W_i(G(\Phi))$; that of $x^0\lambda^{n-i}$ is the Whitney number of the first kind, $w_i(G(\Phi))$. The Tutte polynomial $T_{G(\Phi)}(r,s)$ equals $(s-1)^{-b(\Phi)}Q_\Phi(r-1,s-1)$. So far these properties are obvious or standard. What is new and important is the existence of balanced expansion formulas, analogs of (2). For the Whitney polynomials,

$$w_\Phi(x,\lambda+1)=\sum_{X\subseteq N} w_{\Phi:X}(x,\lambda)x^{\operatorname{rk}(\Phi:X^c)}, \tag{4}$$

proved in **[17]**, Theorem 1.1. As for the dichromatic polynomials,

$$Q_\Phi(u,v)=\sum_{X\subseteq N}(v+1)^{l_0+|E:X^c|}v^{|X|-n}Q^b_{\Phi:X}\left(u-\frac{1}{v},v\right), \tag{5}$$

proved algebraically in **[14]**.

In the totally biased case we have the following analogs of (3b). The balanced Whitney polynomial is

$$w^b_{(\Gamma,\varnothing)}(x,\lambda)=\sum_F x^{|F|}(-1)^{t(F)}\sum_{k=0}^{t(F)} f_k(\Gamma/F)(-\lambda)^k,$$

summed over spanning forests, which simplifies to

$$w^b_{(\Gamma,\varnothing)}(x,\lambda)=(-1)^n\sum_{k=0}^{n} f_k(\Gamma)(-\lambda)^k(1-x)^{n-k} \tag{6}$$

since $f_k(\Gamma/F)$ is the number of k-tree forests $F' \supseteq F$. For the balanced dichromatic polynomial,

$$Q^b_{(\Gamma,\varnothing)}(u, v) = \sum_{k=0}^{n} f_k(\Gamma) u^k. \tag{7}$$

If we interpret (3u) for the bicircular matroid by means of Equation (1) and similarly interpret the Whitney and dichromatic formulas derived from (4) and (6) and from (5) and (7), we have the next theorem. Note that the Whitney polynomial of a matroid G is $w_G(x, \lambda) = \sum_A x^{\operatorname{rk} A} p_{G/A}(\lambda)$ and Crapo's invariant is $\beta(G) = (-1)^{\operatorname{rk} G - 1} p'_G(1)$, whence $\beta(B(G)) = (-1)^{n-1} \chi'_{(\Gamma,\varnothing)}(1)$.

THEOREM 4. *For the bicircular matroid of a finite graph* Γ, *we have*

$$p_{B(\Gamma)}(\lambda) = \lambda^{-t(\Gamma)} \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X|} \sum_{k=0}^{|X|} f_k(\Gamma\colon X)(1-\lambda)^k,$$

$$w_{B(\Gamma)}(x, \lambda) = \lambda^{-t(\Gamma)} \sum_{X \subseteq N} x^{|X^c| - t(\Gamma: X^c)} (-1)^{|X|} \sum_{k=0}^{|X|} f_k(\Gamma\colon X)(1-\lambda)^k (1-x)^{|X|-k},$$

$$T_{B(\Gamma)}(r, s) = s^{l_0} \sum_{X \subseteq N} s^{|E:X^c|} (s-1)^{-|X^c|} \sum_{k=0}^{|X|} f_k(\Gamma\colon X)\left(r - \frac{s}{s-1}\right)^k.$$

Other invariants are

$$W_i(B(\Gamma)) = \sum_{X \subseteq N} (-1)^{t(\Gamma: X^c)} \sum_{k=n-i}^{|X|} \binom{k}{n-i} \binom{|X|-k}{n-i-k-t(\Gamma\colon X^c)} f_k(\Gamma\colon X),$$

$$w_i(B(\Gamma)) = (-1)^i \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} \sum_{k=n-i}^{|X|} \binom{k}{n-i} f_k(\Gamma\colon X)$$

(*the value of the outer sum is positive*),

$$\mu(B(\Gamma)) = (-1)^{n-t(\Gamma)} \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} \sum_{k=t(\Gamma)}^{|X|} \binom{k}{t(\Gamma)} f_k(\Gamma\colon X),$$

$$\beta(B(\Gamma)) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} f_1(\Gamma\colon X).$$

(*Note that* f_1 *is the number of spanning trees.*) *The number of independent sets of size* r *is*

$$I_r = \sum_{X \subseteq N} \sum_{k=0}^{|X|} (-1)^{k+r-n} \binom{|E\colon X^c|}{k+r-|X|} \binom{k}{n-r} f_k(\Gamma\colon X),$$

and the number of bases is

$$I_{n-t(\Gamma)} = \sum_{X\subseteq N} \sum_{k=0}^{|X|} (-1)^{k-t(\Gamma)} \binom{|E: X^c|}{k+|X^c|-t(\Gamma)} \binom{k}{t(\Gamma)} f_k(\Gamma: X).$$

The value of I_r is, from the definition of the dichromatic polynomial, the coefficient of $u^{n-r}v^0$ in $Q_{(\Gamma,\varnothing)}(u, v)$, which can be obtained from (5) and (7). The values in Theorem 4 were obtained by discarding the free loops before doing the calculation. The fact that the coefficient of v raised to a negative power is 0 imposes some mild constraints on the forest numbers $f_k(\Gamma: X)$.

The formulas can be rewritten as sums over all spanning forests in the form $\sum_F (\cdots) \sum_X (\cdots)$, where X ranges over subsets of the isolated nodes in F. Particularly if every node supports a loop one gets nice formulas of the type found in [**13**]—see the comment following Theorem 10 below.

By considering in the light of the coloring interpretation the value of $\chi^b_{(\Gamma,\varnothing)}(2^{|E|})$ we obtain the following fact.

Corollary 5. *For a finite graph $\Gamma = (N, E)$, let $V = \{0, 1\}^E$ be the verticial hypercube whose coordinates correspond to E. The number of mappings $\kappa: N \to V$ such that for no edge e between v and w is it true that*

$$\kappa(v)_e \neq \kappa(w)_e, \quad \text{and} \quad \kappa(v)_f = \kappa(w)_f \quad \text{for all} \quad f \neq e,$$

is equal to

$$\sum_{k=0}^{n} (-1)^{n-k} 2^{k|E|} f_k(\Gamma).$$

Working directly with the lattice of flats leads to another formula for the Möbius invariant, obtained by inverting the next result over the set of those $X \subseteq N$ without tree components. Let $\mu^*(\Gamma)$ denote the Möbius invariant of the cographic matroid $G^*(\Gamma)$.

Theorem 6. *The Möbius invariants of the bicircular matroids of induced subgraphs of $\Gamma = (N, E)$ satisfy the formula*

$$\sum_X \mu(B(\Gamma: X)) \cdot (-1)^{|X^c|-c(\Gamma:X^c)} |\mu^*(\Gamma: X^c)| = 0 \tag{8}$$

unless all edges are free loops, where X ranges over node sets for which $t(\Gamma: X) = 0$.

Proof. Assume, without loss of generality, that there are no free loops. Write the Möbius function $\mu(0, F)$ of a forest $F \in \mathscr{F}_0$, the lattice described in Theorem 3, in the form

$$\mu(0, F) = \mu(B[\Gamma: N(F)^c])(-1)^{|E(F)|},$$

and split the defining recurrence for μ into a double sum,

$$\sum_F \mu(0, F) = \sum_X \mu(B(\Gamma\colon X)) \sum_{F:\, N(F)=X^c} (-1)^{|E(F)|}.$$

The left-hand side equals 0 unless $\operatorname{rk} B(\Gamma) = 0$, that is $\Gamma = (N, \varnothing)$. The right-hand side reduces to the sum in (8).

2c. THE FOREST LATTICE. Theorem 3 shows that the forest lattice of Γ is the lattice of flats of $\mathcal{B}(\Gamma^\circ)$. Hence we have the facts stated in the introduction. The formulas of Theorem 4 simplify because the only stable node set is the empty set and $t(\Gamma\colon Y) = 0$ for every $Y \subseteq N$.

THEOREM 7. *For the forest lattice $\mathcal{F}$ of a finite, loopless graph Γ, we have*

$$p_{\mathcal{F}}(\lambda) = (-1)^n \sum_{k=0}^{n} f_k(\Gamma)(1-\lambda)^k,$$

$$\begin{aligned} w_{\mathcal{F}}(x, \lambda) &= \sum_{X \subseteq N} x^{|X^c|}(-1)^{|X|} \sum_{k=0}^{|X|} f_k(\Gamma\colon X)(1-\lambda)^k(1-x)^{|X|-k} \\ &= x^n \sum_{F \in \mathcal{F}} \left(\frac{\lambda-1}{x-1}\right)^{t(F)} \left(\frac{x-1}{x}\right)^{|N(F)|}, \end{aligned}$$

$$T_{\mathcal{F}}(r, s) = \left(\frac{1}{s-1}\right)^n \sum_{X \subseteq N} s^{|E:X^c|}(s-1)^{|X|} \sum_{k=0}^{|X|} f_k(\Gamma\colon X)\left(r - \frac{s}{s-1}\right)^k,$$

and in addition

$$W_i(\mathcal{F}) = \sum_{X \subseteq N} (-1)^{n-|X|} \sum_{k=0}^{|X|} \binom{k}{n-i}\binom{|X|-k}{n-k-i} f_k(\Gamma\colon X),$$

$$w_i(\mathcal{F}) = (-1)^i \sum_{k=n-i}^{n} \binom{k}{n-i} f_k(\Gamma),$$

$$\mu(\mathcal{F}) = (-1)^n \sum_{k=0}^{n} f_k(\Gamma),$$

$$\beta(\mathcal{F}) = f_1(\Gamma).$$

In addition we have a strengthening of Theorem 6. (A similar but decidedly more complicated result exists for the general bicircular matroid and even more generally for biased graphs [**14**].) Let Π_N be the set of partitions of N; let $\Gamma\colon \pi$ denote the induced subgraph $\bigcup_{B \in \pi} (\Gamma\colon B)$.

THEOREM 8. *The Möbius invariant of the forest lattice of a graph $\Gamma = (N, E)$ with $N \neq \varnothing$ satisfies*

$$\mu(\mathcal{F}) = (-1)^n \sum_{\pi \in \Pi_N} (-1)^{c(\Gamma:\pi)-|\pi|}(|\pi|)! \cdot |\mu^*(\Gamma\colon \pi)|.$$

Proof. Let $f(X)=\mu(B(\Gamma\colon X))$ and $g(Y)=(-1)^{|Y|-c(\Gamma\colon Y)}\,|\mu^*(\Gamma\colon Y)|$ for $X, Y\subseteq N$. Then from (8) applied to $\Gamma\colon Z$ we have

$$\sum_{X\subseteq Z} f(X)g(Z\backslash X)=\delta(\varnothing, Z).$$

(Note that $f(\varnothing)=g(\varnothing)=1$.) This relation can be solved for f: we have

$$f(X)=\sum_{\pi\in\Pi_N}(-1)^{|\pi|}(|\pi|)!\prod_{B\in\pi} g(B),$$

from which the theorem follows.

3. Three geometric realizations

An arrangement $\mathcal{H}$ of finitely many hyperplanes in real linear or affine space dissects the space into pieces of various dimensions, called the *faces* of $\mathcal{H}$; in particular the components of the complement of $\bigcup\mathcal{H}$ are the *regions* of the arrangement. With voltage graphs we can calculate the numbers of regions and faces of an arrangement whose hyperplanes are determined by *two-term* equations: equations of the forms $x_j=ax_i$, $x_i=a$ (with $a\neq 0$), $x_i=0$, and $0=0$ (the latter defining the "degenerate hyperplane," a necessary technicality). Such a "two-term" arrangement has the most faces when it is in a kind of general position which corresponds to a totally biased voltage graph and hence to the bicircular matroid.

The number of regions of $\mathcal{H}$ we denote $o(\mathcal{H})$; the number of i-dimensional faces is $o_i(\mathcal{H})$.

3a. HOMOGENEOUS ARRANGEMENTS. A two-term hyperplane in real linear space $\mathbb{R}^n$ has an equation in one of the forms $x_j=ax_i$, $x_i=0$, or $0=0$. From the equations of $\mathcal{H}$ we can construct in the following way its *graph* Γ and, by labelling the edges, its *voltage graph* Φ. The graphs have the node set $N=\{v_1, v_2, \ldots, v_n\}$. For each hyperplane $x_j=ax_i$ $(a\neq 0)$ there is an edge e_{ij} with voltage $\varphi(e_{ij})=a$; for a hyperplane $x_i=0$ there is a loop e_{ii} at v_i with any voltage $\varphi(e_{ii})\neq 1$; for a degenerate hyperplane $0=0$ there is a free loop. Notice that while Φ determines $\mathcal{H}$ completely, Γ determines only the variables appearing in the equations of $\mathcal{H}$, not the exact hyperplanes.

The *rank* of $\mathcal{S}\subseteq\mathcal{H}$ is

$$\operatorname{rk}\mathcal{S}=n-\dim\left(\bigcap\mathcal{S}\right). \tag{9}$$

This rank function determines a matroid structure on $\mathcal{H}$. It is easy to see by comparing independent sets (by for example [**15**], Theorem 5.1 (c), adapted to voltage graphs) that the correspondence $\mathcal{H}\leftrightarrow E(\Phi)$ is a matroid isomorphism. Thus we need not be fussy about distinguishing sharply between $\mathcal{H}$ and Φ.

The question we address here is that of the *maximum* numbers of regions and faces if each hyperplane is allowed to shift slightly while maintaining its two-termedness. That lets the voltages vary but forces the graph Γ to remain the same. It is clear from geometry (or more abstractly, by (10) below, from the work of Lucas on weak maps of matroids: [**5**], p. 130, and [6], Table 7.2) that generalizing the position of an arrangement by reducing the dimension of intersection of some hyperplanes will increase the number of faces. What is not immediately obvious is that there is any one most general position for a two-term arrangement $\mathcal{H}$ with a specified graph Γ. That however follows by examining a simple bound on the rank of $\mathcal{S}$. Clearly $\operatorname{rk} \mathcal{S} \leq n(\mathcal{S})$, the number of coordinates involved in the equations of hyperplanes in $\mathcal{S}$. Moreover any additional hyperplane can reduce the dimension by at most 1. Thus

$$\operatorname{rk} \mathcal{S} \leq \min_{\mathcal{T} \subseteq \mathcal{S}} (n(\mathcal{T}) + |\mathcal{S} \setminus \mathcal{T}|). \tag{$*$}$$

If it is possible to achieve equality in $(*)$, then there is a most general position for $\mathcal{H}$ given Γ and, comparing $(*)$ with Theorem 1 (j′), that position is the one in which $G(\mathcal{H}) = B(\Gamma)$. But of course equality can be achieved. For instance let φ be a voltage on Γ in which each $\varphi(e)$ is a different prime number; then Φ can have no balanced circles, so $G(\Phi) = B(\Gamma)$. In fact it is more than sufficient to choose the voltages so the numbers $\log|\varphi(e)|$ are linearly independent over the rationals.

So we can calculate the maximum numbers of faces by applying Theorem A of [**11**] (or see [**12**]), which states that the numbers of regions and faces of $\mathcal{H}$ are obtained from the characteristic and Whitney polynomials of $G(\mathcal{H})$ by the formulas

$$o(\mathcal{H}) = (-1)^{\operatorname{rk} \mathcal{H}} p_{G(\mathcal{H})}(-1), \tag{10r}$$

$$\sum_{i=0}^{n} o_i(\mathcal{H}) x^{n-i} = (-1)^{\operatorname{rk} \mathcal{H}} w_{G(\mathcal{H})}(-x, -1). \tag{10f}$$

We get the maximum numbers by taking the matroid $G(\mathcal{H})$ to be $B(\Gamma)$ and using Theorem 4.

THEOREM 9. *An arrangement of two-term hyperplanes in $\mathbb{R}^n$ with graph Γ has at most the following numbers of regions and i-dimensional faces:*

$$o(\mathcal{H}) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} \sum_{k=0}^{|X|} 2^k f_k(\Gamma : X),$$

$$o_i(\mathcal{H}) = \sum_{X \subseteq N} (-1)^{t(\Gamma : X^c)} \sum_{k=0}^{|X|} 2^k \binom{|X| - k}{i - k - t(\Gamma : X^c)} f_k(\Gamma : X).$$

These bounds are attained by any generally positioned two-term arrangement $\mathcal{H}$ with the given graph Γ.

These formulas are especially simple when $\mathcal{H}$ contains all the coordinate hyperplanes, for then $\varnothing$ is the only stable node set and $t(\Gamma: X^c) \equiv 0$. The set of flats of such an arrangement, ordered by reverse inclusion, is isomorphic to the forest lattice of Γ.

3b. AFFINE ARRANGEMENTS. The equations of two-term equations in real affine space $\mathbb{E}^d$ are of the forms $x_j = ax_i$, $x_i = a$, $x_i = 0$, and $0 = 0$. The best way to treat them is to introduce homogeneous coordinates $(x_0, x_1, \ldots, x_d)$ and the ideal hyperplane h_∞: $x_0 = 0$. Then an affine arrangement $\mathcal{A}$ corresponds to a homogeneous arrangement $\mathcal{H}$ in $\mathbb{R}^{d+1}$ with one more hyperplane, $x_0 = 0$. The region and face numbers of $\mathcal{H}$ can be computed by Theorem 9 and those of $\mathcal{A}$ calculated by the formulas

$$o(\mathcal{A}) = (\mathcal{H}), \tag{11r}$$

$$o_i(\mathcal{A}) = o_{i+1}(\mathcal{H}) - o_{i+1}(\mathcal{H}_\infty), \tag{11f}$$

where $\mathcal{H}_\infty$ denotes the arrangement in h_∞ induced by the equations of $\mathcal{A}$. Let Γ be the graph of $\mathcal{H}$, whose node set is $\{v_0, v_1, \ldots, v_d\}$. We call it also the graph of $\mathcal{A}$ and the voltage graph of $\mathcal{H}$ we call that of $\mathcal{A}$.

The voltage graph of $\mathcal{H}_\infty$ is computable from Φ. But since we are here interested in maxima and in general position we omit the general definition. All we need is that $G(\mathcal{H}_\infty) = G(\Phi)/e_0$, where e_0 is the loop at v_0. Thus when $\mathcal{H}$ is general, $G(\mathcal{H}_\infty) = B(\Gamma)/e_0 = B(\Gamma_{e_0})$ by Theorem 2.

THEOREM 10. *An arrangement of two-term hyperplanes in affine space $\mathbb{E}^d$, with graph Γ on the nodes $N = \{v_0, v_1, \ldots, v_d\}$, has at most the following numbers of regions and faces:*

$$o(\mathcal{A}) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} \sum_{k=0}^{|X|} 2^k f_k(\Gamma: X),$$

$$o_i(\mathcal{A}) = \sum_{\substack{X \subseteq N \\ v_0 \in X}} (-1)^{t(\Gamma: X^c)} \sum_{k=0}^{|X|} 2^k \binom{|X| - k}{i + 1 - k - t(\Gamma: X^c)} f_k(\Gamma: X).$$

These bounds are attained by any generally positioned two-term affine arrangement $\mathcal{A}$ with the given graph Γ.

Note that general position for an affine arrangement $\mathcal{A}$ means its homogeneous version is general, equivalently that the voltage graph Φ is totally biased. This rules out certain parallelisms in $\mathcal{A}$. The criterion is not that each set of affine hyperplanes have minimal affine intersection.

The proof is by substituting in (11) using Theorem 9, then simplifying the formula for $o_i(\mathscr{A})$ by noting that, if $v_0 \in Y \subseteq N$, the induced subgraphs Γ: Y and Γ_{e_0}: Y have the same numbers of tree components.

The numbers of bounded regions and faces can also be calculated provided that the smallest flats of $\mathscr{A}$ are points. (In an affine arrangement we do not regard $\varnothing$ as a flat.) An equivalent condition is that $\operatorname{rk} \mathscr{H} = d+1$. Let $b(\mathscr{A})$ and $b_i(\mathscr{A})$ be the numbers of bounded regions and i-faces. By Theorem D of [**11**] (or see [**12**]) we have

$$b(\mathscr{A}) = \beta(G(\mathscr{H})). \tag{12r}$$

To get $b_i(\mathscr{A})$ we use the fact that each bounded i-face is a bounded region of the arrangement induced by $\mathscr{A}$ on one of its i-flats, and conversely. Thus since $G(\mathscr{H}) = B(\Gamma)$ in the general case,

$$b_i(\mathscr{A}) = \sum_A \beta(B(\Gamma)/A), \tag{12f}$$

the sum being taken over all i-dimensional flats A, or equivalently over all closed sets A in $B(\Gamma)$ such that $e_0 \notin A$ and $\operatorname{rk} A = d-i$.

For a forest F, let $p(F)$ be the number of ways to add links among the trees of F to make F itself a tree. Let $q_i(\Gamma\colon Y)$ be the number of sets $X \subseteq Y$ for which Γ: X is a i-tree forest and $t(\Gamma\colon (Y \backslash X)) = 0$.

THEOREM 11. *An arrangement of two-term affine hyperplanes with graph Γ has at most the following numbers of bounded regions and faces:*

$$b(\mathscr{A}) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} f_1(\Gamma\colon X),$$

$$b_i(\mathscr{A}) = \sum_{\substack{T \text{ tree} \\ v_0 \in N(T)}} \sum_{k=0}^{|N(T)|} (-1)^{i+1-k} q_{i+1-k}(\Gamma\colon N(T)^c) f_k(T).$$

These maxima are attained if $\mathscr{A}$ lies in general position.

Proof. The first equation is easy: it results from substituting into (12) from Theorem 4.

To prove the second equation we begin with the description of flats given in Theorem 3. We can regard the flats A in (12) as the forests such that $v_0 \in N(A)$ (whence $e_0 \notin A$) and $t(\Gamma\colon N(A)^c) = 0$. Thus spanning trees in the contraction $B(\Gamma)/A$ can be regarded as linkings of A into a tree in Γ with the same node set $N(A)$, and the expression given by Theorem 4 for $\beta(B(\Gamma)/A) = \beta(B(\Gamma_A))$ can be read as the sum

$$\sum_F (-1)^{t(A \backslash F)} p(F),$$

ranging over all subforests (that is, unions of trees) of A such that Γ: $N(A\backslash F)=A\backslash F$. Thus $v_0\in N(F)$. Note that $d-i=\operatorname{rk} A=n-t(A)$, where $n=|N|=d+1$. Summing over A for which $t(A)=i+1$ we have

$$b_i(\mathscr{A})=\sum_{\substack{F\text{ forest}\\ v_0\in N(F)}}(-1)^{i+1-t(F)}p(F)q_{i+1-t(F)}(\Gamma\colon N(F)^c).$$

The formula of the theorem follows easily.

As usual the results simplify considerably when $\mathscr{A}$ contains all the coordinate hyperplanes. The only node set with a stable complement is N; $t(\Gamma\colon X^c)\equiv 0$; and $q_j(\Gamma\colon Y)=\delta_{0j}$. Thus the maxima are

$$b(\mathscr{A})=f_1(\Gamma), \tag{13r}$$

$$b_i(\mathscr{A})=\sum_{k=i}^{d}\binom{k}{i}(\#\text{ of }(k+1)\text{-node trees meeting }v_0) \tag{13f}$$

3c. BARYCENTRIC ARRANGEMENTS. The barycentric version of a two-term arrangement is perhaps the most interesting. We choose $d+1$ affinely independent points $p_0, p_1, \ldots, p_d$ in $\mathbb{E}^d$, forming the vertices of a geometric simplex s^d, and introduce the barycentric coordinates $(x_0, x_1, \ldots, x_d)$ subject to $\sum x_i=1$, in which p_i has coordinates $(0,\ldots,0,1_i,0,\ldots,0)$. Two-term hyperplanes have equations of the forms $x_i=0$ and $x_j=ax_i$. The former is the facet hyperplane of s^d opposite p_i. The latter is the hyperplane spanned by

$$p=\frac{1}{a+1}p_i+\frac{a}{a+1}p_j$$

and all the vertices except p_i and p_j. We call this an *apical hyperplane* and p the *apex*. If $a>0$ we call p an *internal* apex since then it is on an edge of s^d; otherwise p is *external*. If all apices are internal and all facet hyperplanes are present in the arrangement, we have an apical dissection of the simplex of the type treated in [**1**], [**2**], and [**13**].

The graph Γ and voltage graph Φ of a barycentric arrangement $\mathscr{H}$ are determined from its equations just as for homogeneous arrangements. But now there is more to the story: it matters how $\mathscr{H}$ is related to the ideal hyperplane h_∞: $\sum x_i=0$. In fact one can obtain this relationship from Φ (see Theorem 13); but to find the maximum numbers of faces that is not necessary, for the maxima obviously occur when h_∞ is generally positioned with respect to $\mathscr{H}$. Then to attain the largest values $\mathscr{H}$ itself must be in general position, which means, as usual, that $G(\mathscr{H})=B(\Gamma)$.

Some extra notation: The *projective completion* of $\mathscr{H}$ is the arrangement $\mathscr{H}'=\{h'\colon h\in\mathscr{H}\}$, where h' denotes the projective subspace generated by the affine flat h. We denote by $\mathscr{H}_\infty$ the induced ideal arrangement: $\mathscr{H}_\infty=\{h_\infty\cap h'\colon h\in\mathscr{H}\}$.

Suppose that h_∞ lies generally with respect to $\mathscr{H}$. That means it contains no flat of H', except $\varnothing$ if $\bigcap \mathscr{H}' = \varnothing$. Hence when the latter holds good, that is when $\operatorname{rk} \mathscr{H}' = d+1$, we have by Corollaries 2.1 and 2.2 of **[11]** the formulas

$$o(\mathscr{H}) = (-1)^{d+1}[p_{G(\mathscr{H}')}(-1) - p_{G(\mathscr{H}')}(0)], \tag{14r}$$

$$\sum_{i=0}^{d} o_i(\mathscr{H})x^{d-i} = (-1)^{d-1}[w_{G(\mathscr{H}')}(-x,-1) - w_{G(\mathscr{H}')}(-x,0)], \tag{14f}$$

and for the bounded regions and faces

$$b(\mathscr{H}) = (-1)^{d+1}\mu(G(\mathscr{H}')), \tag{15r}$$

$$\sum_{i=0}^{d} b_i(\mathscr{H})x^{d-i} = (-1)^{d+1}w_{G(\mathscr{H}')}(-x,0) - x^{d+1}. \tag{15f}$$

When on the other hand $\operatorname{rk} \mathscr{H}' \leq d$, $\mathscr{H}'$ is homogeneous (whence $G(\mathscr{H}') = G(\mathscr{H})$) so our Theorem 9 applies. Thus we assume henceforth that $\mathscr{H}'$ has rank $d+1$.

Now to get the largest numbers of faces we assume Φ is totally biased, that is $G(\mathscr{H}') = B(\Gamma)$. It follows that to have rank $d+1$ we must have $t(\Gamma) = 0$.

THEOREM 12. *Let $\mathscr{H}$ be a barycentric arrangement of two-term hyperplanes in $\mathbb{E}^d$ whose graph Γ has no tree components. The maximum numbers of regions and faces (total, bounded, and unbounded) are*

$$o(\mathscr{H}) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} \sum_{k=0}^{|X|} (2^k - 1) f_k(\Gamma : X),$$

$$o_i(\mathscr{H}) = \sum_{X \subseteq N} (-1)^{t(\Gamma : X^c)} \sum_{k=0}^{|X|} (2^k - 1) \binom{|X| - k}{i - k - t(\Gamma : X^c)} f_k(\Gamma : X);$$

$$b(\mathscr{H}) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} (-1)^{|X^c|} \sum_{k=0}^{|X|} f_k(\Gamma : X),$$

$$b_i(\mathscr{H}) = \sum_{X \subseteq N} (-1)^{t(\Gamma : X^c)} \sum_{k=0}^{|X|} \binom{|X| - k}{i - k - t(\Gamma : X^c)} f_k(\Gamma : X);$$

and for the unbounded region and face numbers $u(\mathscr{H})$ and $u_i(\mathscr{H})$, the same formulas as for $o(\mathscr{H})$ and $o_i(\mathscr{H})$ except with $2^k - 1$ replaced by $2^k - 2$. These maxima are attained when $\mathscr{H}$ is generally positioned both within itself and with respect to the ideal hyperplane.

The formulas simplify markedly when all the facet hyperplanes of the simplex are included in $\mathscr{H}$ (the case corresponding to the forest lattice of Γ): then only $\varnothing$ is stable, and $t(\Gamma : X^c) \equiv 0$. Here it is easy to eliminate the sums over X, yielding formulas like Equations (1)–(3) in [**13**]. When in addition all apices are internal (i.e., all voltages are positive), the bounded faces are precisely those in s^d; this fact makes possible a much more detailed analysis like the one carried out in [**1**] and in [**13**] (see especially Theorem 2).

As remarked in [**13**], p. 254, the total number of regions or i-faces has the form (something simple) − (bounded). The explanation lies in how $\mathscr{H}$ compares to the homogeneous arrangement in $\mathscr{H}_{\mathbb{R}}$ in $\mathbb{R}^{d+1}$ with the same equations. The barycentric arrangement $\mathscr{H}$ is the cross section of $\mathscr{H}_{\mathbb{R}}$ by the hyperplane $\sum x_i = 1$. The faces missed by the cross section are precisely those diametrically opposite to faces whose sections are bounded. Hence $o(\mathscr{H}) = o(\mathscr{H}_{\mathbb{R}}) - b(\mathscr{H})$, $u(\mathscr{H}) = o(\mathscr{H}_{\mathbb{R}}) - 2b(\mathscr{H})$, and so forth. The "something simple", in other words, is just the number of regions or i-faces of a homogeneous arrangement, in conformity with the general principle that homogeneous arrangements are simpler than affine ones.

The question of just when h_∞ lies in general position with respect to $\mathscr{H}$ can be answered precisely. In the voltage graph Φ let (X, R) be a balanced, connected subgraph. Pick one node $v \in X$ and for each $w \in X$ let $A(v, w)$ be the product of voltages along a path from v to w, including $A(v, v) = 1$. The product is path-independent because R is balanced. Let

$$A_R(v) = \sum_{w \in X} A(v, w).$$

Since $A_R(w) = A(w, v)A_R(v)$, whether $A_R(v)$ is zero or not is independent of the choice of v; we write $A_R = 0$ or $A_R \neq 0$ accordingly.

THEOREM 13. *Let $\mathscr{H}$ be a barycentric arrangement of two-term hyperplanes in $\mathbb{E}^d$. A necessary and sufficient condition for $\mathscr{H}$ to be in general position with respect to h_∞ is that there be no spanning subgraph of Φ for which each balanced component (X, R) has $A_R = 0$.*

To prove the theorem we ask when a flat of $\mathscr{H}_{\mathbb{R}}$, say $t = \bigcap \mathscr{S}_{\mathbb{R}}$, lies in the hyperplane h_0: $\sum x_i = 0$. The points in t are the points for which $x_j = \varphi(e)x_i$ for every edge $e \in S$ (the edge set corresponding to $\mathscr{S}_{\mathbb{R}}$) from v_i to v_j. These equations force $x_i = 0$ for each v_i in an unbalanced component of (N, S). In a balanced component (X, R) the constraints simplify to $x_j = A(v_i, v_j)x_i$ for all $v_j \in X$, where v_i is one chosen element of X. Thus for $x \in t$,

$$\sum x_i = \sum_{(X,R)} A_R(v_i)x_i,$$

summed over the balanced components (X, R) of (N, S). The only way all $x \in t$ can be guaranteed to lie in h_0 is by having all $A_R(v_i) = 0$. That is the claimed criterion.

If, say, $X=\{v_0, v_1, \ldots, v_k\}$, we can write the condition on (X, R) in the form

$$\sum_{j=1}^{k} A(v_0, v_j) \neq -1. \tag{16}$$

If $\mathscr{H}$ is an apical dissection of the simplex s^d—that is, if all voltages are positive—then (16) can never be satisfied. So for such arrangements the problem of generality with respect to the ideal hyperplane is no problem. The contrary case is that corresponding to signed graphs, where all voltages are $+1$ or -1, and in particular to all-negative signed graphs. Then for instance the existence of a single negative link, say e, such that Φ: $N(e)^c$ has no balanced components, means $\mathscr{H}$ does *not* meet h_∞ in general position.

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