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A CODING APPROACH TO SIGNED GRAPHS*

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Abstract. The cocycle code of an undirected graph Γ is the linear span over $\mathbf{F}_2$ of the characteristic vectors of cutsets. (If Γ is complete bipartite, this is the generalized Gale–Berlekamp code.) The natural bijection between the cosets of this code and the switching classes of signed graphs based on Γ is used to show that the number of such classes is equal to the number of even-degree subgraphs of Γ in both the labeled and unlabeled cases and to improve by coding theory previous bounds on $D(\Gamma)$, the maximum line index of imbalance of signings of Γ. Bounds on $D(\Gamma)$ are obtained in terms of the genus of Γ and on the number of unlabeled even-degree subgraphs in terms of $D(\Gamma)$. Numerous examples are treated, including the "grid" (or "lattice") graphs that are of interest in the Ising model of spin glasses.

Key words. signed graph, line index of imbalance, frustration, cutset code, cocycle code, covering radius, genus, spin glass, Ising model, Gale–Berlekamp code, switching class, even-degree subgraph

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1. Introduction.

Graphs with signed edges, called *signed graphs,* were introduced in the 1950s [12] in connection with a problem in attitudinal psychology [7] and have since often been rediscovered, notably in the theory of spin glasses [31]. The fundamental feature of a signed graph is the list of circuits that are *positive,* that is, have an even number of negative edges. We call two signed graphs *switching equivalent* if they have the same base graph and the same positive circuits. The equivalence classes (called *switching classes*) of signings of a graph correspond to the additive cosets of the binary cocycle space of Γ; by considering this space as a linear code (the *cocycle code* or *cutset code*), we can use coding theory to obtain results on two problems about signed graphs.

A signed graph is called *balanced* if every cycle is positive. One measure of the degree to which a signed graph fails to be balanced is the smallest number of edges whose negation produces a balanced signed graph. This quantity is known as the *line index of imbalance* or *frustration index.* (The former name is from [13]; the concept, in different formulations, originates in [1] and [13] and later from spin glass theory [3, §2.2]—whence comes the term "frustration"—and possibly elsewhere.) The question is how large the index can be, given Γ. We obtain new upper and lower bounds on the maximum index, improving the results of [2], by observing that it equals the covering radius of the cutset code.

Using an approach to imbalance in signed planar graphs first developed in [15] and our recent results on the covering radius of cycle codes [29], we obtain lower bounds on the line index of imbalance of a signed graph based on a graph of known genus.

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The number of switching classes of signings of a graph Γ is known to equal the number of even-degree subgraphs of Γ. This is trivial if Γ is labeled, a theorem of Wells if Γ is unlabeled [32]. We observe that Wells's theorem is precisely the application to cutset codes of a standard automorphism property of a linear code. Then we use the cutset code to obtain a lower bound on the number of unlabeled even-degree subgraphs.

2. The cutset code of a graph. We let Γ denote a finite, loopless graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma) = \{e_1, e_2, \ldots, e_m\}$. We write $n = |V|, m = |E|$, and $c =$ the number of connected components of Γ. A *cocycle* or *cutset* is an edge set that consists of all edges having one endpoint in some set $X \subseteq V$ and one endpoint not in X. Under the operation of set sum (i.e., symmetric difference), the cutsets form a subspace of the binary vector space of all subsets of E. Replacing subsets S of E by their characteristic vectors $x_S \in \mathbf{F}_2^m$ gives a subspace of $\mathbf{F}_2^m$, which we call the *cutset* or *cocycle code*, $C^*(\Gamma)$. The dual code is the *cycle code* $C(\Gamma)$, defined as the linear span of the characteristic vectors of circuits of Γ. As is well known, both codes have length m; their dimensions are $n - c$ for the cocycle code and $m - n + c$ for the cycle code.

The *cosets* of a linear code C are the additive cosets of C in $\mathbf{F}_2^m$. The *weight* $w(x)$ of an element x of $\mathbf{F}_2^m$ is the number of 1s it contains. The *distance* from x to y in $\mathbf{F}_2^m$ is $d(x, y) = w(x - y)$. The *minimum weight* of a coset $x + C$ is $\min_{y \in C} w(x + y)$. The *covering radius* $R(C)$ is the largest minimum weight of a coset taken over all cosets.

3. Signed graphs and switching classes. A *signed graph* is a pair (Γ, σ), where σ is a function from the edges of Γ into $\{-1, +1\}$. A signed graph (Γ, σ') is said to be obtained by *switching* (Γ, σ) at $X \subseteq V$ if

$$\sigma'(xy) = \begin{cases} \sigma(xy) & \text{if } x, y \in X \text{ or } x, y \in V \backslash X, \\ -\sigma(xy) & \text{if } x \in X \text{ and } y \notin X \text{ or } x \notin X \text{ and } y \in X, \end{cases}$$

where xy denotes an edge whose endpoints are x and y. Two signed graphs are said to be *switching equivalent* if they are based on the same graph and are exchanged by switching at some X. It is easy to see that switching equivalence is an equivalence relation. It is also easy to show that two signings of Γ are switching equivalent if and only if they have the same positive circuits (see [33, Prop. 3.2], for instance).

Switching is derived ultimately from [16], where it was employed to study two-graphs, which are, in effect, signed graphs (K_n, σ) on the complete graph K_n. The precise definition is as follows: a *two-graph* on n vertices is a class of three-element subsets of an n-set V such that each four-element subset of V contains an even number of members of the class. The original article on two-graphs is [30]. An excellent exposition of this line of development is [24].

A simple but crucial lemma is the following.

LEMMA 1. *The mapping* $+1 \to 0$, $-1 \to 1$ *induces one-to-one correspondences* (i) *between signed graphs based on* Γ *and elements of* $\mathbf{F}_2^m$, *and* (ii) *between switching classes of signed graphs based on* Γ *and cosets of* $C^*(\Gamma)$.

Proof. (i) is obvious. (ii) Switching at X in $(+1, -1)$ notation is equivalent to adding in (0, 1) notation the cocycle determined by X and its complement. Thus a switching class is, in $(0, 1)$ notation, a coset of $C^*(\Gamma)$. □

4. Covering radius and index of imbalance. It is clear that a signed graph is balanced if and only if it is switching equivalent to the all-plus signed graph $(\Gamma, +1)$. The smallest number $d(\Gamma, \sigma)$ such that (Γ, σ) can be changed into a balanced graph by changing the sign of $d(\Gamma,\sigma)$ edges is called the *line index of imbalance* of (Γ,σ). The largest possible line index of imbalance over all signings of a graph Γ, call it $D(\Gamma)$, was investigated in [2]. These authors proved inter alia that

$$\frac{m}{2} - \sqrt{nm} \leq D\left(\Gamma\right) \leq \frac{m}{2} \tag{1}$$

and

$$D\left(K_{t,t}\right) \leq \frac{t^2}{2} - c_0 t^{3/2} \tag{2}$$

for some constant c_0 that can be taken as $c_0 = \pi/480$.

We can improve these results by using known facts on the covering radius of codes (some of which were originally expressed in terms of ± 1 matrices). Theorem 1 improves the lower bound in (1). Our proof is, moreover, conceptually simpler than the proof of (1). Formula (6) sharpens (2), and Theorem 2 shows that the quantity $m/2 - D(\Gamma)$ can be bounded away from zero more generally than is done in (2).

THEOREM 1. *It is the case that*

$$\frac{m}{2} - \sqrt{\frac{\ln 2}{2}}\sqrt{m\left(n-c\right)} \leq D\left(\Gamma\right). \tag{3}$$

We mention that (3) is derived in [26] from a slightly stronger formula, which in our notation is

$$D\left(\Gamma\right) \geq mH^{-1}\left(1 - \frac{n-c}{m}\right), \tag{4}$$

where H is the binary entropy function (H_2 in [17, pp. 308–309]).

THEOREM 2. *If* Γ *is simple and bipartite,*

$$D\left(\Gamma\right) \leq \frac{m - \sqrt{m}}{2}. \tag{5}$$

THEOREM 3. [5, p. 266, Thm.]. *We have*

$$\frac{t^2}{2} - \frac{t^{3/2}}{2} + o\big(t^{3/2}\big) \leq D(K_{t,t}) \leq \frac{t^2}{2} - \frac{t^{3/2}}{\sqrt{2\pi}} + o\left(t^{3/2}\right). \tag{6}$$

The error term in the upper bound of Theorem 3 was reduced to $O(t^{1/2})$ in [9, Cor. 3 and Rem.].

For the proofs, we first need a lemma.

LEMMA 2.

(i) *The line index of imbalance of a signed graph based on* Γ *is equal to its minimum weight as a coset of* Γ.

(ii) $D(\Gamma)$ *is equal to the covering radius of* $C^*(\Gamma)$.

Proof. (i) is clear by translating $(+1,-1)$ notation into $(0,1)$ notation. (ii) is clear from (i), since the covering radius of a code is the largest possible minimum weight of a coset. □

Proof of Theorem 1. For any $[N,K]$ code, a simple consequence of the sphere-covering bound is $R \geq N/2 - \sqrt{\frac{1}{2}NK\ln 2}$ (stated in [20]; the proof is given in [26, Thm. 2]). Here $N = m, K = n-c$, and $R = D(\Gamma)$. □

Proof of Theorem 2. Furthermore, if a binary code is of strength 2 and contains the all-one vector, then $R \leq (N-\sqrt{N})/2$ [14, Thm. 3]. Here the strength is 2 because the minimum distance of the cycle code is ≥ 3. $C^*(\Gamma)$ contains the all-one vector if and only if Γ is bipartite. □

Note that, for any graph, the bound $D(\Gamma) \leq m/2$ of [2] is also implied by the results of [14].

Proof of Theorem 3. We indicate how the quantity evaluated in [5], which is the minimum number of -1s in line negations of a ± 1-matrix (of order $p \times q$, say), maximized over all such matrices, equals $D(K_{p,q})$. Let $V(K_{p,q}) = X \cup Y$, where $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_q\}$. An entry a_{ij} in the matrix is the sign of the edge $x_i y_j$. Switching a vertex, say x_i, corresponds to negating a line, in this case, row i. □

Interest in $D(K_{p,q})$, although not by that name, was aroused by the Gale–Berlekamp switching game [5], [8]–[10] and the corresponding binary code, which happens to be $C^*(K_{10,10})$. The code $C^*(K_{p,q})$ is usually described as the p by q array code whose codewords are sums of matrices of the form

$$\begin{bmatrix} & 1 & \\ 0 & \vdots & 0 \\ & 1 & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & 0 & \\ 1 & \cdots & 1 \\ & 0 & \end{bmatrix}.$$

This is a $[pq, p+q-1]$ code, which might be called a *generalized Gale–Berlekamp code* because it corresponds to the p by q Gale–Berlekamp switching game. Identifying rows (respectively, columns) with the first set (respectively, second set) of the bipartition of $K_{p,q}$, we see that the code is $C^*(K_{p,q})$. Indeed, switching at a point in $K_{10,10}$ is the same as switching a commutator in the Gale–Berlekamp switching game.

Theorems 1 and 3 suggest that the best general asymptotic lower bound on $D(\Gamma)$ has the form

$$\frac{m}{2} - c_1\sqrt{mn} + o\left(\sqrt{mn}\right) \leq D\left(\Gamma\right),$$

where c_1 is a positive constant lying between $\sqrt{\frac{1}{2}\ln 2} \approx .59$ and $1/2\sqrt{\pi} \approx .28$. We would like to know the exact value of c_1, both for all graphs and for narrower classes, especially for classes in which m/n^2 is bounded away from zero and for the class of k-connected graphs, where $k \geq 1$.

For the latter, the graphs $K_{p,q}$ with $k \leq p \leq q$ are relevant. It is known [9, Thm. 3], [10, p. 396, Ex. b] that

$$D\left(K_{p,q}\right) \leq \frac{pq}{2} - \frac{pq}{2^p}\begin{pmatrix} p-1 \\ \lfloor \frac{p-1}{2} \rfloor \end{pmatrix} \approx \frac{1}{\sqrt{2\pi}}\sqrt{mn} \quad \text{if } q >> p. \tag{7}$$

Exact values are known when $p \leq 5$. It is easy to prove that

$$D\left(K_{1,q}\right) = 0, \quad D\left(K_{2,q}\right) = \left\lfloor \frac{q}{2} \right\rfloor, \quad D\left(K_{3,q}\right) = \left\lfloor \frac{3q}{4} \right\rfloor$$

for all q. It is true, but difficult to prove, that

$$D\left(K_{4,q}\right) = \left\lfloor \frac{5}{4}q \right\rfloor - \alpha_4,$$

where $\alpha_4 = 0$, except $\alpha_4 = 1$ if $q \equiv 1,4,5 \pmod 8$, and that

$$D(K_{5,q}) = \left\lfloor \frac{25}{16} q \right\rfloor - \alpha_5,$$

where $\alpha_5 = 0$, except $\alpha_5 = 1$ if $q \equiv 2,4,9,13,15 \pmod{16}$. (The formulas for $p \le 4$ were published in [5] without proof. That for $p = 5$ is new.) We conclude that, as $q \to \infty$ while p is fixed,

$$D(K_{p,q}) = \frac{pq}{2} - b_p\sqrt{mn} + O(1),$$

where $b_1 = \frac{1}{2}, b_2 = \sqrt{2}/4 \approx .35, b_3 = \sqrt{3}/4 \approx .43, b_4 = \frac{3}{8} = .375$, and $b_5 = 3\sqrt{5}/16 \approx .42$. (The values approach $1/\sqrt{2\pi} \approx .40$, by (7).)

The exact values $D(K_{t,t})$ are also known, for $t \le 10$. They are published in [8, Table 1] as the values R_t of the covering radius of the Gale–Berlekamp codes $C^*(K_{t,t})$.

Concerning Theorem 2, we conjecture that a somewhat similar upper bound holds for all simple graphs.

CONJECTURE 1. $D(\Gamma) \le m/2 - (n-c)/4$.

The conjectured bound is sharp since it is essentially equal to D for K_n, since $D(K_n) = \lceil (n-1)/2)^2 \rceil$. (This was stated implicitly in [1, Thm. 14]. It was proved in [21].) Conjecture 1 is stronger than Theorem 2 for most bipartite graphs, including all those in which $q \ge p + 2\sqrt{p} + 1$.

For graphs with large girth, stronger upper bounds than those of (1) and Theorem 2 follow from the generalized Norse bounds of [28]. We let $\mu_k(m)$ denote the kth centered moment of a $B(m, \frac{1}{2})$ binomial probability distribution. Some small values, taken from [28], are

$$\mu_2(m) = \frac{m}{2^2}, \quad \mu_4(m) = \frac{3m^2 - 2m}{2^4}, \quad \mu_6(m) = \frac{15m^3 - 30m^2 + 16m}{2^6}.$$

THEOREM 4. *If Γ has girth $g \ge 2s+2$, where s is a positive integer, then*

$$D(\Gamma) \le \frac{m}{2} - \left(2^{1-1/(2s+1)} - 1\right) \mu_{2s}(m)^{1/2s}.$$

If Γ is bipartite, then

$$D(\Gamma) \le \frac{m}{2} - \mu_{2s}(m)^{1/2s}.$$

Proof. The proof follows from [28, Thms. 6 and 1]. □

When $s = 1$, the latter part of this theorem is Theorem 2.

The next result is an asymptotic version of Theorem 4.

THEOREM 5. *If Γ has girth $g \ge 2s+2$, where s is a fixed positive integer, then, as $m \to \infty$, we have*

$$D(\Gamma) \le \frac{m}{2} - \frac{\sqrt{m}}{2}\left(2^{1-1/(2s+1)} - 1\right)\left(\frac{(2s)!}{s!}\right)^{1/2s} + o\left(\sqrt{m}\right)$$

and, if Γ is bipartite,

$$D(\Gamma) \le \frac{m}{2} - \frac{\sqrt{m}}{2}\left(\frac{(2s)!}{s!}\right)^{1/2s} + o\left(\sqrt{m}\right).$$

Proof. The proof follows from [28, Thms. 7 and 3]. □

For general graphs with fixed minimum girth, better estimates are attainable. For instance, from [28, Thms. 4 and 5], we obtain Theorem 6.

THEOREM 6. *If* Γ *has girth* ≥ 4, *then*

$$D(\Gamma) \leq \frac{m}{2} - f_1 \frac{\sqrt{m}}{2},$$

where $f_1 \approx .68$.

If Γ *has girth* $g \geq 6$, *then*

$$D(\Gamma) \leq \frac{m}{2} - h_2 \mu_4 (m)^{1/4},$$

where $h_2 \approx .82$.

The exact values of f_1 and h_2 can be obtained from [28].

The bounds of Theorem 6 are stronger than those of Theorem 4 evaluated at $s = 1$ and 2. However, they are still probably rather weak. (Since they are so general, applying to all binary codes, that is not surprising. The same remark applies to all general results of this section, although they are the best known bounds on $D(\Gamma)$.) For instance, the Petersen graph P has girth 5. Theorem 6 gives $D(P) \leq 5.7$, but it can be shown that $D(P) = 3$. For an example falling under Theorem 2 (i.e., the case where $s = 1$ and Γ is bipartite of Theorem 4), see Example 10.

5. Embedded graphs. In this section, all graphs are connected. For many planar graphs $\Gamma, D(\Gamma)$ is easy to compute because of duality. Let Γ^* be the planar dual of Γ. Then $C^*(\Gamma) = C(\Gamma^*)$; hence $D(\Gamma) = R(C(\Gamma^*))$. However, this equals $\lfloor \frac{1}{2} n^* \rfloor$ (where $n^* = |V(\Gamma^*)| =$ the number of faces of a planar embedding of Γ, that is, $m + 2 - n$) for many graphs, for instance, if Γ^* is Hamiltonian or if it is connected, k-regular, and k-edge-connected [29].

As far as we know, the first to apply planar duality to find the imbalance of signed graphs were Katai and Iwai [15, §4]. They used minimum T-joins in Γ^*, for suitable T, to calculate $d(\Gamma, \sigma)$. (T-joins and their connection to the cycle code are explained in [29].) Previously, $d(\Gamma, -1)$ had been obtained similarly in [19], where it was regarded as the largest cut size in the plane graph Γ.

Example 1. $K_{2,q}$ is planar. Its dual graph is C_q with doubled edges, which is Hamiltonian. We recover the result of §4, $D(K_{2,q}) = \lfloor q/2 \rfloor$.

Example 2. A *wheel* W_n is the join of a circuit C_{n-1} and a vertex. It is planar and self-dual. Because it is Hamiltonian, $D(W_n) = \lfloor n/2 \rfloor$.

Example 3. A *biwheel* B_n is the join $C_{n-2} \vee \bar{K}_2$, where $\bar{K}_p$ is the edgeless graph of order p. Its planar dual is a *circular ladder* CL_{n-2}, which consists of two vertex-disjoint C_{n-2}'s, say with vertex sets $\{x_1, x_2, \ldots, x_{n-2}\}$ and $\{y_1, y_2, \ldots, y_{n-2}\}$, and edges $x_i y_i$ for all i. This is Hamiltonian (and cubic and three-edge-connected), so we have $D(B_n) = n - 2$.

Example 4. Dually, $D(CL_n) = \lfloor n/2 \rfloor + 1$ since B_{n+2} is Hamiltonian.

Example 5. The *ladder* L_n is CL_n with the edges $x_1 x_n$ and $y_1 y_n$ deleted. We see that $D(L_n) = \lfloor n/2 \rfloor$, since its dual contains the Hamiltonian spanning subgraph $P_{n-1} \vee K_1$, where P_{n-1} is a path of order $n - 1$. The latter (a *fan*) is self-dual; hence $D(P_{n-1} \vee K_1) = \lfloor n/2 \rfloor$, as well.

Example 6. The *rectangular grid* $G_{p,q}$ is of some interest as a finite spin-glass model [31], [3]. It has vertex set $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ and edge set $\{(i_1, j_1)(i_2, j_2) : |i_1 - i_2| + |j_1 - j_2| = 1\}$. So $n = pq$ and $m = 2pq - p - q$. Its planar dual $G^*_{p,q}$ is $G_{p-1,q-1}$ with an extra vertex adjacent to every outer vertex of $G_{p-1,q-1}$. Since $G^*_{p,q}$ is Hamiltonian and has $(p-1)(q-1)+1$ vertices,

$$D(G_{p,q}) = \left\lfloor \frac{pq-p-q+2}{2} \right\rfloor.$$

The bounds given by Theorems 1 and 2 are far from the true value. Theorem 1, for instance, approximately gives $D > pq(1-\sqrt{\ln 2}) \approx .17pq$ for $G_{p,q}$.

We can generalize this work to nonplanar graphs as follows: Let χ denote the largest Euler characteristic of any surface in which Γ embeds and χ_0 the maximum over orientable embedding surfaces. We call $\gamma = \frac{1}{2}(2-\chi_0)$ the *genus* and $\hat{\gamma} = 2-\chi$ the *demi-genus* of Γ. A simple use of the Euler formula yields the following lemma.

LEMMA 3. *Let Γ^* denote the dual of Γ as cellularly embedded in a compact surface of demi-genus d. Then*

$$\dim C(\Gamma^*) - \dim C^*(\Gamma) = d.$$

Proof. Subtract $\dim C(\Gamma^*) = m - n^* + 1$ from $\dim C(\Gamma) = n-1$ and apply Euler's formula $n - m + n^* = 2 - d$. □

Using simple results of coding theory yields the following result. We denote the covering radius of $C(\Gamma)$ by $\tau(\Gamma)$.

THEOREM 7. *We have the bounds*

$$\left\lfloor \frac{n(\Gamma^*)}{2} \right\rfloor \le \tau(\Gamma^*) \le D(\Gamma) \quad \text{and} \quad \tau(\Gamma^*) \ge \left\lfloor \frac{D(\Gamma)}{1+\hat{\gamma}} \right\rfloor.$$

Proof. $C^*(\Gamma) \subseteq C(\Gamma^*)$, hence their covering radii satisfy the reversed relationship $\tau(\Gamma^*) \le D(\Gamma)$. However, $\tau(\Gamma^*) \ge \lfloor n(\Gamma^*)/2 \rfloor$ by [29]. The second lower bound on $\tau(\Gamma^*)$ follows from Lemma 3 and the theorem of Simonis [25]. □

COROLLARY 1. $D(\Gamma) \ge \lfloor 1 - \hat{\gamma}/2 + (m-n)/2 \rfloor$.

Proof. Substitute from Euler's formula into the first bound of Theorem 7. □

For fixed $\hat{\gamma}$, this bound is better than Theorem 1, as the following examples (taken from [4]) show.

Example 7. *The Heawood graph.* $n = 14, m = 21$, and the genus is 1. We may take $\hat{\gamma} = 2$. Corollary 1 yields $D \ge 3$, while Theorem 1 yields $D \ge 2$.

Example 8. *The Franklin graph.* $n = 12, m = 18$, and we may take $\hat{\gamma} = 2$. This gives a lower bound of 3. Theorem 1 only gives 1.

Example 9. The r-dimensional *hypercube graph* Q_r has $n = 2^r$ and $m = 2^{r-1}r$. By Theorems 1 and 2,

$$2^{r-2}r\left(1 - \sqrt{\frac{4\ln 2}{r}}\right) < D(Q_r) \le 2^{r-2}r - \sqrt{2^{r-3}r}. \tag{8}$$

(Here $4\ln 2 = 2.772\ldots$.)

Computing $D(Q_r)$ exactly is surely hard, but we know

$$D(Q_1) = 0, \quad D(Q_2) = 1, \quad D(Q_3) = 3.$$

The values for $r \le 2$ are obvious. Q_3 is planar, and Q_3^* is the octahedral graph. The latter is Hamiltonian, so $D(Q_3) = \tau(Q_3^*) = \lfloor \frac{1}{2}n^* \rfloor = 3$. (From (8), we only obtain $1 \le D(Q_3) \le 4$.)

We also know that the minimum $\hat{\gamma}$ for Q_r is $2+2^{r-2}(r-4)$ if $r \geq 3$ ([23]; see [11, Thm. 3.5.8]). Corollary 1 then gives the lower bound

$$D(Q_r) \geq 2^{r-3}r = \frac{m}{4}. \tag{9}$$

For instance, $8 \leq D(Q_4) \leq 13$, which is a wide range of uncertainty, but less than that allowed by (8). It is interesting that (9) holds with equality for $r = 2$ and 3. Formula (8) shows this cannot be true for $r \geq 12$.

Example 10. The *toroidal grid* $G'_{p,q}$ has been used as a spin glass model [31], [3]. It consists of $G_{p,q}$ (Example 6), together with "wraparound" edges $(i,1)(i,q)$ and $(1,j)(p,j)$. It has $n = pq, m = 2pq$, and $\hat{\gamma} = 2$. Thus, by Corollary 1, $D(G'_{p,q}) \geq \lfloor pq/2 \rfloor$. Theorem 7 gives the same result, since G'_{pq} is self-dual in the torus, and it is Hamiltonian so $\tau(G'_{p,q}) = \lfloor pq/2 \rfloor$.

On the other hand, if $\Gamma_1 \subseteq \Gamma_2$, clearly $D(\Gamma_2) \leq D(\Gamma_1) + m_2 - m_1$. Taking $G_{p,q} \subseteq G'_{p,q}$, our best estimate for the toroidal grid is therefore

$$\left\lfloor \frac{pq}{2} \right\rfloor \leq D(G'_{p,q}) \leq \left\lfloor \frac{pq+p+q+2}{2} \right\rfloor.$$

Exactly where the true value lies we do not know. It is at neither extreme, since we can show that

$$D\left(G'_{1,q}\right) = q+1 \quad \text{and} \quad D\left(G'_{2,q}\right) = q+2.$$

Theorems 1 and 2 give very poor estimates, similar to those for the rectangular grid in Example 6.

6. Switching classes and even-degree subgraphs. Now we turn to our second problem. An *even-degree subgraph* of Γ is a spanning subgraph in which all degrees are even. In other words, it is a spanning subgraph whose edge set is an element of the cycle space $C(\Gamma)$, the dual code of $C^*(\Gamma)$. It is easy to see the next theorem by comparing the dimensions of $C(\Gamma)$ and $\mathbf{F}_2^m/C^*(\Gamma)$.

THEOREM 8. *For a labeled graph* Γ, *the number of switching classes of signings of* Γ *equals the number of even-degree subgraphs.*

What happens in the unlabeled case, specifically if we let the automorphism group of Γ act on the cosets of $C^*(\Gamma)$ and the words of $C(\Gamma)$? Call two subgraphs of Γ, or (switching classes of) signings, Γ-*isomorphic* if one is carried to the other by an automorphism of Γ. Then we have the following result.

THEOREM 9. (see [32]). *The number of* Γ-*isomorphic even-degree subgraphs of* Γ *is equal to the number of* Γ-*isomorphic switching classes of signed graphs based on* Γ.

This result was proved for $\Gamma = K_n$ in [18], then in a cohomological context in [6], and later generalized to arbitrary Γ, again with a cohomological proof [32, Thm. 1.3].

Proof. We apply to the cycle code the well-known fact [22, p. 211] that a group acting on the coordinate places of a linear code C has as many orbits on the codewords of C as on the cosets of its dual. □

We now use imbalance to obtain a lower bound on the number in Theorem 9.

THEOREM 10. *The number of* Γ-*isomorphic even-degree subgraphs of* Γ *is at least equal to* $D(\Gamma) + 1$.

Proof. Since two Γ-isomorphic signed graphs based on Γ correspond to equivalent cosets of $C^*(\Gamma)$, they have the same line index of imbalance. Hence, there are at least $D(\Gamma) + 1$ signed graphs based on Γ that are Γ-nonisomorphic and, from Theorem 9, at least that many even-degree subgraphs in Γ. □

COROLLARY 2. *There are at least* $\lfloor (n-1)^2/4 \rfloor + 1$ *two-graphs on* n *vertices.*

Proof. From [21], we know that $D(K_n) = \lfloor (n-1)^2/4 \rfloor$. □

Call a binary linear code *completely transitive* [27] if its covering radius equals the number of orbits of cosets under the action of its automorphism group (not counting the code itself).

COROLLARY 3. *If* $D(\Gamma) + 1$ *equals the number of* Γ*−isomorphic even-degree subgraphs of* Γ, *then* $C^*(\Gamma)$ *is completely transitive.*

Proof. From Theorem 9 and Lemma 1, $D(\Gamma) + 1$ counts the number of Γ-isomorphism classes of cosets of $C^*(\Gamma)$. However, every automorphism of Γ induces a coordinate automorphism of $C^*(\Gamma)$. □

Example 11. Consider $C^*(K_4)$. Note that $K_4 = W_4$. Hence, from §5 or [21], we know that $D(K_4) = 2$. All even subgraphs of K_4 are isomorphic to either $\bar{K}_4, K_3$, or C_4. Thus we obtain a completely transitive [8, 3] code with covering radius 2.

Example 12. $C^*(K_{3,3})$ is completely transitive. A direct proof is given in [27]. All even subgraphs are isomorphic to either $\bar{K}_6, C_4$, or C_6. It is known from [5] (and easily proved) that $D(K_{3,3}) = 2$. So we have a completely transitive [9, 5] code with covering radius 2.

7. Conclusion and open problems. In this paper, we have shown how coding theory could be used to generalize certain enumerative results on two-graphs and to improve certain estimates on the line index of imbalance of signed graphs. The next question is to see if graph theory can be of some use in designing good new covering codes or decoding classical ones.

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